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Second-order wave diffraction by a submerged circular cylinder

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Expressions are derived for the amplitudes of the second-harmonic waves generated when a uniform wave train is normally incident upon a two-dimensional body, submerged in water of infinite depth. These amplitudes are given in terms of integrals over the free surface of products of first-order quantities. For a submerged, circular cylinder, it is shown analytically that there is no second-order reflected wave at any frequency. This extends the classical result that there is no reflection at first-order for this body.

1. Introduction

In recent years, there has been a great deal of interest expressed in the calculation of inviscid, second-order wave loading on fixed and floating bodies. Maruo (1960) showed that second-order theory predicted a mean drift force on a body which did not arise if linear theory only was used. This force, whilst small in magnitude, could cause significant motion of a body over a large period of time. Another second-order phenomenon is the slowly varying force on a body, which results from the interaction of two linear wave trains of similar frequency. Such forces are discussed in Newman (1974). The frequencies at which the forces occur are often close to the natural frequencies of moored bodies and so can lead to resonant oscillations of the body. The total second-order force on a body, due to a monochromatic wave train, is comprised of several components. Pinkster (1979) has shown that some of these components may be calculated directly from products of first-order quantities, whilst others depend on the second-order potential. It is, however, unnecessary to calculate the full second-order wave field around a body simply in order to calculate that part of the force due to the second-order potential. Molin (1979) applied Green's theorem to the second-order diffraction potential and the first-order potential due to the body oscillating at twice the frequency of the incoming wave, and obtained an expression for this part of the force as an integral of products of first-order quantities.

More recently, Vada (1987) numerically solved the first- and second-order diffraction problems for a submerged cylinder of arbitrary shape, in two dimensions. He was then able to calculate the first- and second-order forces on the body by direct integration of the pressure around the body surface. In addition to determining the forces on a body, he calculated the second-order reflection and transmission coefficients, given by R_2 and T_2 respectively, directly from the second-order potential. In particular, he observed that the magnitude of R_2 for a submerged, circular cylinder was of the same order as the accuracy in his numerical scheme. This gives rise to the interesting speculation that R_2 is identically zero for a submerged, circular cylinder, which will in fact be proved in this work. The remarkable result that at first-order there is zero reflection from a submerged, circular cylinder is well-known and due to Dean (1948). The theoretical result that R_2 is zero is supported by the experimental evidence of Chaplin (1984) who observed that there is no detectable reflection at second or third order.

In this paper, formulae for the second-order reflection and transmission coefficients associated with an arbitrary two-dimensional body in infinite depth are derived. Green's theorem is used to relate R_2 and T_2 to the first-order, double-frequency diffraction potential. Similar formulae for R_2 and T_2 have been derived in concurrent work by Wu (1990). He used them to calculate numerically the second-order reflection and transmission from horizontal cylinders. His numerical results for the submerged cylinder are consistent with the present work. In the final part of the paper, the second-order reflection coefficient for a submerged, circular cylinder is shown to be zero, for all frequencies of the incoming wave.

2. The second-order reflection coefficient

A fixed, two-dimensional body with surface denoted by S_B, is totally submerged in water of infinite depth. Rectangular Cartesian coordinates are chosen such that, under linear theory, the origin is in the mean free surface and the y-axis points vertically downwards, as illustrated in figure 1. A uniform wave train is incident on the body, travelling in the direction of decreasing x. The fluid is assumed to be inviscid and incompressible and the motion irrotational. Thus, the flow is described by a velocity potential Φ which satisfies Laplace's equation in the fluid. Neglecting surface tension, the kinematic and dynamic boundary conditions at the free surface are given by

$$\frac{\partial \Phi}{\partial y}-\frac{\partial \zeta}{\partial t}-\frac{\partial \Phi}{\partial x}\frac{\partial \zeta}{\partial x}=0 \quad \text{on} \quad y=\zeta \tag{2.1}$$

and

$$\frac{\partial \Phi}{\partial t}+\tfrac{1}{2}(\nabla \Phi)^2-g\zeta=0 \quad \text{on} \quad y=\zeta \tag{2.2}$$

respectively, where $y=\zeta(x,t)$ is the wave elevation and g is the acceleration due to gravity. There is no normal flow through the body surface and so the potential satisfies

$$\frac{\partial \Phi}{\partial n}=0 \quad \text{on} \quad S_B, \tag{2.3}$$

where n is a normal coordinate to the surface. The fluid is at rest at large depths, thus

$$\nabla \Phi \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty. \tag{2.4}$$

In addition to the boundary conditions (2.1)–(2.4), the form of the wave field as $x \rightarrow \pm\infty$ must be specified, in order to fully determine Φ.

The wave steepness ϵ is assumed to be small and so the velocity potential and wave elevation may be expanded as

$$\Phi = \epsilon\Phi_1+\epsilon^2\Phi_2+O(\epsilon^3) \tag{2.5}$$

and

$$\zeta = \epsilon\zeta_1+\epsilon^2\zeta_2+O(\epsilon^3). \tag{2.6}$$

The potentials Φ_1 and Φ_2 individually satisfy Laplace's equation and the boundary

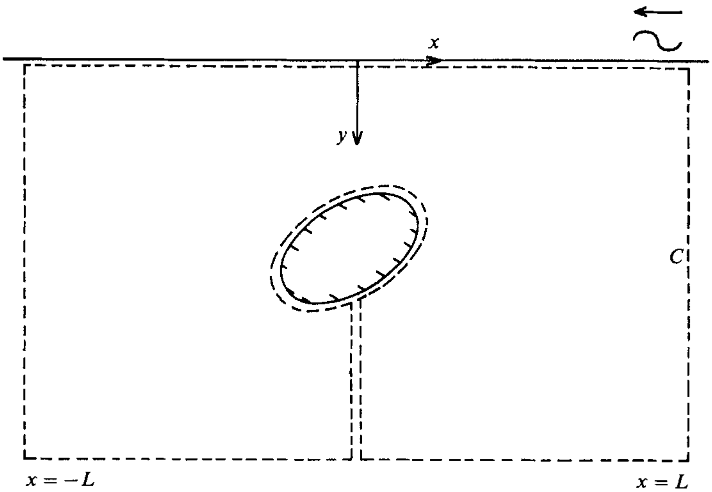

FIGURE 1. Definition sketch.

conditions (2.3) and (2.4). At first order, the kinematic and dynamic boundary conditions on the free surface are combined to give

$$\frac{\partial^2 \Phi_1}{\partial t^2} - g\frac{\partial \Phi_1}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{2.7}$$

whilst the free-surface boundary condition for the second-order potential is

$$\frac{\partial^2 \Phi_2}{\partial t^2} - g\frac{\partial \Phi_2}{\partial y} = -\frac{\partial}{\partial t}[(\nabla\Phi_1)^2] - \frac{1}{g}\frac{\partial \Phi_1}{\partial t}\frac{\partial}{\partial y}\left[\frac{\partial^2 \Phi_1}{\partial t^2} - g\frac{\partial \Phi_1}{\partial y}\right] \quad \text{on} \quad y = 0. \tag{2.8}$$

The incident first-order wave has amplitude A and frequency ω and so the total first-order potential is written as

$$\Phi_1(x,y,K,t) = \mathrm{Re}\left[\frac{-\mathrm{i}gA}{\omega}\varphi_1(x,y,K)\,\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{2.9}$$

where
$$K = \frac{\omega^2}{g}. \tag{2.10}$$

The time-independent potential satisfies the far-field condition

$$\varphi_1(x,y,K) \sim \begin{array}{ll} [\mathrm{e}^{-\mathrm{i}Kx} + R_1(K)\,\mathrm{e}^{\mathrm{i}Kx}]\,\mathrm{e}^{-Ky}, & x \to +\infty, \\ T_1(K)\,\mathrm{e}^{-\mathrm{i}Kx}\,\mathrm{e}^{-Ky}, & x \to -\infty, \end{array} \tag{2.11}$$

where $R_1(K)$ and $T_1(K)$ are the first-order reflection and transmission coefficients respectively. Using the form of Φ_1 given by (2.9), the second-order free-surface boundary condition, (2.8), may be rewritten as

$$\frac{\partial^2 \Phi_2}{\partial t^2} - g\frac{\partial \Phi_2}{\partial y} = \mathrm{Re}\,[F(x)\,\mathrm{e}^{-2\mathrm{i}\omega t}] + F_s(x) \quad \text{on} \quad y = 0, \tag{2.12}$$

where
$$F(x) = -\frac{g^2A^2}{\omega^2}\left[\mathrm{i}\omega(\nabla\varphi_1)^2 - \tfrac{1}{2}\mathrm{i}\omega\varphi_1\frac{\partial}{\partial y}\left[K\varphi_1 + \frac{\partial \varphi_1}{\partial y}\right]\right] \quad \text{on} \quad y = 0 \tag{2.13}$$

and
$$F_s(x) = \frac{\mathrm{i}g^2A^2}{4\omega}\left[\varphi_1\frac{\partial^2 \varphi_1^*}{\partial x^2} - \varphi_1^*\frac{\partial^2 \varphi_1}{\partial x^2}\right] \quad \text{on} \quad y = 0, \tag{2.14}$$

where * denotes complex conjugate. The form of (2.12) suggests that Φ_2 may be decomposed as

$$\Phi_2(x,y,t) = \Phi_s(x,y) - \Gamma t + \mathrm{Re}\,[A^2\omega\varphi_2(x,y)\,\mathrm{e}^{-2\mathrm{i}\omega t}], \tag{2.15}$$

where φ_2 and Φ_s also depend on frequency although this is not explicitly stated. The steady and double-frequency parts of the potential, Φ_s and φ_2, individually satisfy Laplace's equation and the boundary conditions (2.3) and (2.4). The free-surface boundary condition (2.12) is split into the two conditions

$$\frac{\partial \Phi_s}{\partial y} = -\frac{F_s(x)}{g} \quad \text{on} \quad y = 0 \tag{2.16}$$

and

$$4K\varphi_2 + \frac{\partial \varphi_2}{\partial y} = f(x) \quad \text{on} \quad y = 0, \tag{2.17}$$

where

$$f(x) = \frac{\mathrm{i}}{K}\left[\left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \tfrac{3}{2}K^2\varphi_1^2 + \tfrac{1}{2}\varphi_1\frac{\partial^2 \varphi_1}{\partial x^2}\right] \quad \text{on} \quad y = 0. \tag{2.18}$$

The choice of Γ in (2.15) affects the position of the mean free surface, but for the purposes of this work, the value of Γ is irrelevant.

The far-field form of φ_2 is determined by considering the asymptotic behaviour of $f(x)$ as $|x| \to \infty$. This in turn depends on the far-field form of φ_1, which is given by (2.11), and it may be shown that

$$f(x) \sim \begin{matrix} 4\mathrm{i}KR_1(K), & x \to +\infty, \\ 0, & x \to -\infty. \end{matrix} \tag{2.19}$$

The form of $f(x)$ for large x suggests that the far-field behaviour of φ_2 is given by

$$\varphi_2 \sim \begin{matrix} \mathrm{i}R_1(K) + R_2\,\mathrm{e}^{4\mathrm{i}Kx-4Ky}, & x \to +\infty, \\ T_2\,\mathrm{e}^{-4\mathrm{i}Kx-4Ky}, & x \to -\infty. \end{matrix} \tag{2.20}$$

When x is large and positive, the first term in (2.20) arises from the interaction of the first-order incident and reflected waves, whilst the second term represents a free wave of frequency 2ω radiating outwards. There is no contribution to the incident wave in water of infinite depth at second order. When x is large and negative, the only term in the representation for φ_2 corresponds to a free wave of frequency 2ω propagating outwards. The second-order reflection and transmission coefficients, R_2 and T_2, are determined by requiring φ_2 to satisfy the body boundary condition (2.3).

It is of interest to calculate R_2 and T_2 because they represent the magnitude of the second-harmonic waves which are generated. These coefficients may be determined without solving the full second-order problem using a method similar to that described by Molin (1979). He showed, using Green's theorem, that the second-order forces on a body due to the second-order potential may be related to the first-order radiation potential due to the body oscillating with frequency 2ω. Using a similar technique, formulae for R_2 and T_2 are generated in terms of the first-order scattering potential at frequency 2ω. Application of Green's theorem to $\varphi_1(x,y,4K)$ and φ_2 around the contour C illustrated in figure 1 gives

$$\int_C \left[\varphi_2\frac{\partial \varphi_1}{\partial n} - \varphi_1\frac{\partial \varphi_2}{\partial n}\right]\mathrm{d}S = 0. \tag{2.21}$$

Contributions to the integral in (2.21) arise only from the line $y = 0$ and the lines $x = \pm L$. If L is allowed to tend to infinity then φ_2 and φ_1 take their far-field forms on the lines $x = \pm L$ and (2.21) yields the following formula for the second-order reflection coefficient:

$$R_2 = \lim_{L\to\infty}\left[-\mathrm{i}\int_{-\infty}^{L}\varphi_1(x,0,4K)f(x)\,\mathrm{d}x+\mathrm{i}R_1(K)\,[R_1(4K)\,\mathrm{e}^{4\mathrm{i}KL}-\mathrm{e}^{-4\mathrm{i}KL}]\right]. \quad (2.22)$$

An expression for the second-order transmission coefficient is derived from a similar application of Green's theorem and

$$T_2 = \lim_{L\to\infty}\left[-\mathrm{i}\int_{-\infty}^{L}\psi_1(x,0,4K)f(x)\,\mathrm{d}x+\mathrm{i}R_1(K)\,T_1(4K)\,\mathrm{e}^{4\mathrm{i}KL}\right], \quad (2.23)$$

where ψ_1 is the first-order potential due to a wave incident on the body from large negative x.

The integrands of the integrals in (2.22) and (2.23) do not, in general, decay as $|x|\to\infty$ and so the integrals must be combined with the other non-zero terms in the expression before the $\lim_{L\to\infty}$ is taken. Equations (2.22) and (2.23) represent formulae for the second-order reflection and transmission coefficients which depend solely on first-order quantities. In the next section, (2.22) will be used to show that the second-order reflection coefficient associated with a horizontal, circular cylinder submerged in water of infinite depth is zero.

3. The submerged circular cylinder

Dean (1948) was the first person to prove the remarkable result that, under linear theory, there is no reflection of a plane wave normally incident on a horizontal, circular cylinder, submerged in water of infinite depth. This result has been extended by Evans (1984), who showed that the magnitude of the amplitude of the far-field waves generated by an oscillating line source in the presence of a cylinder is the same on both sides of the cylinder. Dean's work is extended further in this section where the second-order reflection coefficient is shown to be zero.

The first-order reflection coefficient associated with the submerged, circular cylinder is zero for all frequencies, and so (2.22) for the second-order reflection coefficient simplifies to

$$R_2 = -\mathrm{i}\int_{-\infty}^{\infty}\varphi_1(x,0,4K)f(x)\,\mathrm{d}x. \quad (3.1)$$

The proof that $R_2 = 0$ for all frequencies follows from contour integration. If the integrand in (3.1) is denoted by $W(x)$, it will be shown that $W(z)$, where $z = x+\mathrm{i}y$, is an analytic function in the region $\mathrm{Im}\,[z] \leqslant 0$ and that the asymptotic behaviour of $W(z)$ is such that

$$\left|\int_{C_\Gamma} W(z)\,\mathrm{d}z\right| \to 0 \quad \text{as} \quad \Gamma\to\infty, \quad (3.2)$$

where C_Γ is a semicircle of radius Γ and centre the origin, in the lower half-plane.

Ursell (1950) constructed the first-order diffraction potential for a submerged, circular cylinder from a series of multipole potentials. These are singular solutions of Laplace's equation which satisfy the linear free-surface condition, decay with depth and behave like waves radiating outwards as $|x| \to \infty$. Thorne (1953) showed that the time-independent, symmetric, nth-order multipole potential which has a singularity at the point $(0, h)$ is given by

$$\chi_n^s = \frac{\cos n\theta}{r^n} + \frac{(-1)^{n-1}}{(n-1)!}\int_0^\infty \frac{(K+l)}{(K-l)} l^{n-1}\,\mathrm{e}^{-l(y+h)}\cos lx\,\mathrm{d}l + \frac{(-1)^n}{(n-1)!}2\pi\mathrm{i}K^n\,\mathrm{e}^{-K(y+h)}\cos Kx \quad (3.3)$$

and the corresponding antisymmetric multipole potential has the representation

$$\chi_n^a = \frac{\sin n\theta}{r^n} + \frac{(-1)^n}{(n-1)!}\int_0^\infty \frac{(K+l)}{(K-l)} l^{n-1}\,\mathrm{e}^{-l(y+h)}\sin lx\,\mathrm{d}l + \frac{(-1)^{n-1}}{(n-1)!}2\pi\mathrm{i}K^n\,\mathrm{e}^{-K(y+h)}\sin Kx. \quad (3.4)$$

The polar coordinates (r, θ) are defined such that

$$r\cos\theta = y-h, \quad r\sin\theta = x. \quad (3.5)$$

The details of the construction of the first-order radiation potential for a cylinder of radius a, submerged so that its centre is at a depth h, $(a < h)$, are given by Evans *et al.* (1979). The corresponding linear diffraction potential associated with a wave incident on the cylinder from the right is given by

$$\varphi_1(x, y, K) = \mathrm{e}^{-Ky-\mathrm{i}Kx} + Ka\sum_{n=1}^{\infty}\frac{a^n}{n}p_n\chi_n, \quad (3.6)$$

where

$$\chi_n = \chi_n^s + \mathrm{i}\chi_n^a = \frac{\mathrm{e}^{\mathrm{i}n\theta}}{r^n} + \frac{(-1)^{n-1}}{(n-1)!}\int_0^\infty \frac{(K+l)}{(K-l)} l^{n-1}\,\mathrm{e}^{-l(y+h)-\mathrm{i}lx}\,\mathrm{d}l + \frac{(-1)^n}{(n-1)!}2\pi\mathrm{i}K^n\,\mathrm{e}^{-K(y+h)-\mathrm{i}Kx}. \quad (3.7)$$

The coefficients $\{p_n\}$ satisfy the matrix system of equations

$$p_m + \sum_{n=1}^{\infty} D_{mn}p_n = \frac{(-1)^m}{(m-1)!}(Ka)^{m-1}\,\mathrm{e}^{-Kh}, \quad (3.8)$$

where

$$D_{mn} = \frac{(-1)^{m+n}a^{m+n}}{(m-1)!\,n!}\int_0^\infty \frac{(K+l)}{(K-l)} l^{m+n-1}\,\mathrm{e}^{-2lh}\,\mathrm{d}l + \frac{(-1)^{m+n-1}}{(m-1)!\,n!}2\pi\mathrm{i}(Ka)^{m+n}\,\mathrm{e}^{-2Kh}. \quad (3.9)$$

The particular combination of symmetric and antisymmetric multipole potentials which occurs in (3.6) arises from the application of the boundary condition of no flow through the cylinder surface. This combination leads directly to the result that there is no first-order reflection from the cylinder since

$$\chi_n \sim \begin{cases} \dfrac{(-1)^n}{(n-1)!}4\pi\mathrm{i}K^n\,\mathrm{e}^{-K(y+h)-\mathrm{i}Kx}, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \quad (3.10)$$

and so

$$\varphi_1(x,y,K) \sim \begin{cases} \mathrm{e}^{-Ky-\mathrm{i}Kx}\left[1+4\pi\mathrm{i}\,\mathrm{e}^{-Kh}\sum_{n=1}^{\infty}\frac{(-1)^n (Ka)^{n+1}p_n}{n!}\right], & x\rightarrow-\infty, \\ \mathrm{e}^{-Ky-\mathrm{i}Kx}, & x\rightarrow\infty. \end{cases} \tag{3.11}$$

Ursell (1950) showed that (3.8) has a unique solution such that the series representation for φ_1 in (3.6) converges absolutely and uniformly on the cylinder surface and outside the cylinder in the region $y \geqslant -h'$, where $0 < h' < h$. Thus, using Harnack's theorem on convergence of harmonic series, described in Kellogg (1953, p. 248), it is straightforward to show that the term-by-term differentiated series for $\partial\varphi_1/\partial x$ and $\partial^2\varphi_1/\partial x^2$ are uniformly convergent in the fluid and, in particular, on the line $y = 0$. Furthermore, these series are readily shown to be absolutely convergent. These properties are necessary to justify the manipulations carried out below.

Whittaker & Watson (1935, p. 243) showed that for $y < h$

$$\frac{\mathrm{e}^{\mathrm{i}n\theta}}{r^n} = \frac{(-1)^n}{(n-1)!}\int_0^{\infty} l^{n-1}\,\mathrm{e}^{-l(h-y)-\mathrm{i}lx}\,\mathrm{d}l, \tag{3.12}$$

and so on $y = 0$

$$\varphi_1(x,0,K) = \mathrm{e}^{-\mathrm{i}Kx}+2Ka\sum_{n=1}^{\infty}\frac{a^n(-1)^{n-1}p_n}{n!}\left[\int_0^{\infty}\frac{l^n\,\mathrm{e}^{-lh-\mathrm{i}lx}}{K-l}\mathrm{d}l-K^n\pi\mathrm{i}\,\mathrm{e}^{-Kh-\mathrm{i}Kx}\right]. \tag{3.13}$$

Using a reduction formula, the integral in (3.13) given by

$$I_n = \int_0^{\infty}\frac{l^n\,\mathrm{e}^{-lh-\mathrm{i}lx}}{K-l}\mathrm{d}l \tag{3.14}$$

may be written as
$$I_n = K^n I_0 - \sum_{m=1}^{n}\frac{K^{n-m}(m-1)!}{(h+\mathrm{i}x)^m}, \tag{3.15}$$

where
$$I_0(x,K) = \int_0^{\infty}\frac{\mathrm{e}^{-lh-\mathrm{i}lx}}{K-l}\mathrm{d}l$$

$$= \begin{cases} \mathrm{e}^{-Kh-\mathrm{i}Kx}[-\pi\mathrm{i}-E_1(-Kh-\mathrm{i}Kx)], & x<0, \\ \mathrm{e}^{-Kh}\,\mathrm{Ei}\,(Kh), & x=0, \\ \mathrm{e}^{-Kh-\mathrm{i}Kx}[\pi\mathrm{i}-E_1(-Kh-\mathrm{i}Kx)], & x>0, \end{cases} \tag{3.16}$$

and E_1 and Ei are the exponential integral functions defined in Abramowitz & Stegun (1965, p. 228). This representation for I_0 in terms of the exponential integrals is determined using contour integration. By definition, $E_1(z)$ has a branch cut along the negative real axis and the jump in the function across the cut is defined by

$$E_1(-X\pm\mathrm{i}0) = -\mathrm{Ei}\,(X)\mp\mathrm{i}\pi, \quad X>0, \tag{3.17}$$

and so $I_0(x,K)$ is continuous at $x = 0$. In fact, it is straightforward to show, from the series representation of the exponential integral, that

$$I_0(x,K) = \mathrm{e}^{-K(h+\mathrm{i}x)}\left[\gamma+\ln K(h+\mathrm{i}x)+\sum_{n=1}^{\infty}\frac{(K(h+\mathrm{i}x))^n}{nn!}\right] \tag{3.18}$$

for all real x, where γ is Euler's constant. Thus, I_0 is a continuously differentiable function of x and

$$\frac{\partial I_0}{\partial x} = -\mathrm{i}KI_0 + \frac{\mathrm{i}}{h+\mathrm{i}x}. \tag{3.19}$$

(The same result is obtained by differentiating under the integral sign in (3.16).) Thus, (3.13) becomes

$$\begin{aligned}\varphi_1(x,0,K) &= \mathrm{e}^{-\mathrm{i}Kx}\left[1+2\pi\mathrm{i}\,\mathrm{e}^{-Kh}\sum_{n=1}^{\infty}\frac{(Ka)^{n+1}(-1)^n p_n}{n!}\right] \\ &\quad + 2Ka\sum_{n=1}^{\infty}\frac{a^n(-1)^n p_n}{n!}\sum_{m=1}^{n}\frac{K^{n-m}(m-1)!}{(h+\mathrm{i}x)^m} \\ &\quad + 2I_0(x,K)\sum_{n=1}^{\infty}\frac{(Ka)^{n+1}(-1)^{n-1}p_n}{n!} \\ &= s(K)\,\mathrm{e}^{-\mathrm{i}Kx}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{t_{mn}(K)}{(h+\mathrm{i}x)^m}+\nu(K)\,I_0(x,K).\end{aligned} \tag{3.20}$$

The quantity $f(x)$ in the integral in (3.1) is given by (2.18) and may be rewritten as

$$f(x) = \frac{\mathrm{i}}{K}\left[\frac{\partial\varphi_1}{\partial x}+\mathrm{i}K\varphi_1\right]\left[\frac{\partial\varphi_1}{\partial x}-\mathrm{i}K\varphi_1\right]+\frac{\mathrm{i}\varphi_1}{2K}\left[K^2\varphi_1+\frac{\partial^2\varphi_1}{\partial x^2}\right] \quad \text{on} \quad y=0, \tag{3.21}$$

where $\varphi_1 = \varphi_1(x,0,K)$. Differentiation of (3.13) twice with respect to x gives

$$\begin{aligned}K^2\varphi_1+\frac{\partial^2\varphi_1}{\partial x^2} &= 2Ka\sum_{n=1}^{\infty}a^n(-1)^{n-1}p_n\left[\frac{K}{(h+\mathrm{i}x)^{n+1}}+\frac{(n+1)}{(h+\mathrm{i}x)^{n+2}}\right] \quad \text{on} \quad y=0 \\ &= \sum_{n=1}^{\infty}\frac{q_n}{(h+\mathrm{i}x)^{n+1}}.\end{aligned} \tag{3.22}$$

A similar procedure yields

$$\begin{aligned}\frac{\partial\varphi_1}{\partial x}+\mathrm{i}K\varphi_1 &= 2\mathrm{i}Ka\sum_{n=1}^{\infty}\frac{a^n(-1)^n p_n}{(h+\mathrm{i}x)^{n+1}} \quad \text{on} \quad y=0 \\ &= \sum_{n=1}^{\infty}\frac{r_n}{(h+\mathrm{i}x)^{n+1}}\end{aligned} \tag{3.23}$$

and

$$\frac{\partial\varphi_1}{\partial x}-\mathrm{i}K\varphi_1 = 2\mathrm{i}Ka\sum_{n=1}^{\infty}\frac{a^n(-1)^n p_n}{(h+\mathrm{i}x)^{n+1}}-2\mathrm{i}K\varphi_1 \quad \text{on} \quad y=0. \tag{3.24}$$

Substitution of (3.20), (3.22), (3.23) and (3.24) into (3.1) yields the expression for the second-order reflection coefficient

$$\begin{aligned}R_2 = \int_{-\infty}^{\infty}&\left(s(4K)\,\mathrm{e}^{-4\mathrm{i}Kx}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{t_{mn}(4K)}{(h+\mathrm{i}x)^m}+\nu(4K)\,I_0(x,4K)\right)\left\{\frac{1}{K}\left[\sum_{n=1}^{\infty}\frac{r_n}{(h+\mathrm{i}x)^{n+1}}\right]\right. \\ &\times\left[\sum_{n=1}^{\infty}\frac{r_n}{(h+\mathrm{i}x)^{n+1}}-2\mathrm{i}K\left(s(K)\,\mathrm{e}^{-\mathrm{i}Kx}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{t_{mn}(K)}{(h+\mathrm{i}x)^m}+\nu(K)\,I_0(x,K)\right)\right] \\ &\left.+\frac{1}{2K}\left[\sum_{n=1}^{\infty}\frac{q_n}{(h+\mathrm{i}x)^{n+1}}\right]\left[s(K)\,\mathrm{e}^{-\mathrm{i}Kx}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{t_{mn}(K)}{(h+\mathrm{i}x)^m}+\nu(K)\,I_0(x,K)\right]\right\}\mathrm{d}x.\end{aligned} \tag{3.25}$$

Each of the series in (3.25) is absolutely and uniformly convergent for all x and so term-by-term integration of the expression for R_2 is permitted. From Gradshteyn & Ryzhik (1980, p. 318)

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}bx}}{(h+\mathrm{i}x)^m}\,\mathrm{d}x = 0, \quad b > 0, \quad m > 0, \tag{3.26}$$

and elementary integration gives

$$\int_{-\infty}^{\infty} \frac{1}{(h+\mathrm{i}x)^m}\,\mathrm{d}x = 0, \quad m > 1. \tag{3.27}$$

Thus, the only possible non-zero terms in (3.25) arise from integrals of the form

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}bx} I_0(x,L)}{(h+\mathrm{i}x)^m}\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{I_0(x,K)\,I_0(x,4K)}{(h+\mathrm{i}x)^m}\,\mathrm{d}x, \tag{3.28}$$

where $b \geqslant 0$, $L > 0$ and $m \geqslant 2$. Both these integrals are now shown to be zero.

Whittaker & Watson (1935, p. 91) showed that if a series of functions converges uniformly on a closed contour C and the individual functions are analytic within and on C, then the sum of the series is an analytic function throughout C and its interior. Thus, the function,

$$w(z,L) = \begin{cases} \mathrm{e}^{-L(h+\mathrm{i}z)}\left[\gamma + \ln L(h+\mathrm{i}z) + \sum\limits_{n=1}^{\infty} \dfrac{L(h+\mathrm{i}z)^n}{nn!}\right], & L > 0 \\ \mathrm{e}^{-Lh-\mathrm{i}Lz}[-\pi\mathrm{i} - E_1(-Lh-\mathrm{i}Lz)], & \mathrm{Re}\,[z] < 0, \\ \mathrm{e}^{-L(h-y)}\,\mathrm{Ei}\,(L(h-y)), & \mathrm{Re}\,[z] = 0, \\ \mathrm{e}^{-Lh-\mathrm{i}z}[\pi\mathrm{i} - E_1(-Lh-\mathrm{i}Lz)], & \mathrm{Re}\,[z] > 0 \end{cases} \tag{3.29}$$

is an analytic function of z in any bounded region for which $\mathrm{Im}\,[z] < h$ and has a branch cut along that part of the imaginary axis $\mathrm{Im}\,[z] \geqslant h$. From (3.18)

$$I_0(x,L) = w(x+\mathrm{i}0, L). \tag{3.30}$$

By Cauchy's theorem,

$$\int_C \frac{\mathrm{e}^{-\mathrm{i}bz} w(z,L)}{(h+\mathrm{i}z)^m}\,\mathrm{d}z = \int_C \frac{w(z,K)\,w(z,4K)}{(h+\mathrm{i}z)^m}\,\mathrm{d}z = 0, \tag{3.31}$$

where C is any closed curve in the region $\mathrm{Im}\,[z] \leqslant 0$. The asymptotic behaviour of $w(z,L)$ is determined from that of $E_1(z)$, given by Abramowitz & Stegun (1965), and so

$$w(z,L) = \begin{cases} -\pi\mathrm{i}\,\mathrm{e}^{-Lh-\mathrm{i}Lz} + \dfrac{1}{Lh+\mathrm{i}Lz} + O(|z|^{-2}) & \mathrm{Re}\,[z] < 0, \mathrm{Im}\,[z] \leqslant 0, \\ \dfrac{1}{Lh+\mathrm{i}Lz} + O(|z|^{-2}), & \mathrm{Re}\,[z] = 0, \mathrm{Im}\,[z] \leqslant 0, \\ \pi\mathrm{i}\,\mathrm{e}^{-Lh-\mathrm{i}Lz} + \dfrac{1}{Lh+\mathrm{i}Lz} + O(|z|^{-2}) & \mathrm{Re}\,[z] > 0, \mathrm{Im}\,[z] \leqslant 0. \end{cases} \tag{3.32}$$

This behaviour is sufficient to ensure that

$$\left|\int_{C_r} \frac{\mathrm{e}^{-\mathrm{i}bz} w(z,L)}{(h+\mathrm{i}z)^m}\,\mathrm{d}z\right| \to 0, \quad \left|\int_{C_r} \frac{w(z,K)\,w(z,4K)}{(h+\mathrm{i}z)^m}\,\mathrm{d}z\right| \to 0 \tag{3.33}$$

as $\Gamma \rightarrow \infty$ where $L > 0$, $b \geqslant 0$, $m \geqslant 2$ and C_Γ is a semicircle of radius Γ and centre the origin, in the lower half-plane. By choosing the closed curve C to be that part of the x-axis between $x = -\Gamma$ and $x = \Gamma$ and the curve C_Γ, a combination of (3.30), (3.31) and (3.33) shows that both the types of integrals in (3.28) are zero and so, from (3.25)

$$R_2 = 0. \tag{3.34}$$

4. Discussion

The proof of the main result of the paper, that there is no reflection at second-order from a submerged, circular cylinder, depends crucially on the form of the linear diffraction potential given by equation (3.6). It is the particular combination of symmetric and antisymmetric multipoles in this potential which leads to the result of zero reflection at both first and second order. The actual values of the coefficients $\{p_n\}$, defined by equation (3.8), do not affect the proof. Thus, if it were possible to construct the linear diffraction potential associated with another body from the same combination of multipoles but with different coefficients, there would also be zero reflection from this body at first and second order. It does, however, seem unlikely that another such body exists for which there is no first-order reflection at all frequencies. A variety of bodies do exist for which there is zero first-order reflection at isolated frequencies. At such a frequency, the equation for the second-order reflection coefficient is given by (3.1). An interesting and open question is whether or not zeros of second-order reflection occur at the same frequencies.

In two-dimensional problems, the reflection and transmission properties of the body are related to the horizontal drift force. Maruo (1960) showed that the leading-order contribution to the drift force on a fixed body arises at second-order in the wave steepness ϵ and is proportional to the square of the amplitude of the first-order reflected wave. Thus, for a submerged, circular cylinder there is no contribution to the horizontal drift force at $O(\epsilon^2)$. In general, this drift force may be expressed as a perturbation expansion involving only the even powers of ϵ (the time average of the contributions to the *total* force from the odd powers is zero). Two possible contributions to the horizontal drift force at $O(\epsilon^4)$ are proportional to the squares of the amplitudes of the second-order reflected and transmitted waves. Longuet-Higgins (1977) has shown that, in the absence of a second-order reflected wave, the second-order waves give a negative contribution to the drift force at $O(\epsilon^4)$. This is consistent with the numerical experiments of Cointe (1989) and Dommermuth (1987) who calculated the forces on a submerged cylinder due to the passage of fully nonlinear waves. There is also the possibility of contributions to the drift force at $O(\epsilon^4)$ from interactions between the first-order and third-order wave fields. However, the details of the third-order scattered wave field have not yet been investigated.

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